Exercise 58

If r is a rational function, use Exercise 57 to show that $\lim_{x\to a} r(x) = r(a)$ for every number a in the domain of r.

Solution

Suppose that r is a rational function, specifically a ratio of an nth-degree polynomial in x and an mth-degree polynomial in x.

$$r(x) = \frac{p_1(x)}{p_2(x)}$$

$$= \frac{c_0 + c_1x + c_2x^2 + \cdots + c_nx^n}{d_0 + d_1x + d_2x^2 + \cdots + d_mx^m}$$

Take the limit of $p(x)$ as $x \to a$.

$$\lim_{x\to a} r(x) = \lim_{x\to a} \frac{c_0 + c_1x + c_2x^2 + \cdots + c_nx^n}{d_0 + d_1x + d_2x^2 + \cdots + d_mx^m}$$

The limit of a quotient is the quotient of the limits, provided that the limit of the denominator is not zero.

$$\lim_{x\to a} r(x) = \frac{\lim\limits_{x\to a}(c_0 + c_1x + c_2x^2 + \cdots + c_nx^n)}{\lim\limits_{x\to a}(d_0 + d_1x + d_2x^2 + \cdots + d_mx^m)}$$

The limit of a sum is the sum of the limits.

$$\lim_{x\to a} r(x) = \frac{\lim\limits_{x\to a} c_0 + \lim\limits_{x\to a} c_1x + \lim\limits_{x\to a} c_2x^2 + \cdots + \lim\limits_{x\to a} c_nx^n}{\lim\limits_{x\to a} d_0 + \lim\limits_{x\to a} d_1x + \lim\limits_{x\to a} d_2x^2 + \cdots + \lim\limits_{x\to a} d_mx^m}$$

The limit of a constant times a function is the constant times the limit of the function.

$$\lim_{x\to a} r(x) = \frac{c_0 \lim\limits_{x\to a} 1 + c_1 \lim\limits_{x\to a} x + c_2 \lim\limits_{x\to a} x^2 + \cdots + c_n \lim\limits_{x\to a} x^n}{d_0 \lim\limits_{x\to a} 1 + d_1 \lim\limits_{x\to a} x + d_2 \lim\limits_{x\to a} x^2 + \cdots + d_m \lim\limits_{x\to a} x^m}$$

The limit of a product is the product of the limits.

$$\lim_{x\to a} r(x) = \frac{c_0 \lim\limits_{x\to a} 1 + c_1 \lim\limits_{x\to a} x + c_2 \left(\lim\limits_{x\to a} x\right)^2 + \cdots + c_n \left(\lim\limits_{x\to a} x\right)^n}{d_0 \lim\limits_{x\to a} 1 + d_1 \lim\limits_{x\to a} x + d_2 \left(\lim\limits_{x\to a} x\right)^2 + \cdots + d_m \left(\lim\limits_{x\to a} x\right)^m}$$

$$= \frac{c_0(1) + c_1(a) + c_2(a)^2 + \cdots + c_n(a)^n}{d_0(1) + d_1(a) + d_2(a)^2 + \cdots + d_m(a)^m}$$

$$= \frac{c_0 + c_1a + c_2a^2 + \cdots + c_na^n}{d_0 + d_1a + d_2a^2 + \cdots + d_ma^m}$$

$$= \frac{p_1(a)}{p_2(a)}$$

$$= r(a)$$